

RESEARCH ARTICLE

A Solution to Optimal Consensus of Multi-Agent Systems

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ABSTRACT

This paper aims to solve the optimal consensus problem for homogeneous systems of general linear dynamics and a cost function. Specifically, we present a new method rooted in LQ optimal control theory and observer design. This method is fundamentally different from the existing consensus control algorithms, which rely on nonzero eigenvalues of the communication topology to determine the distributed controller. The analytical solution for the distributed controller is derived via Riccati equations and observers, which is parallel to the classical optimal control theory. Theoretical analysis and a simulation example demonstrate that the proposed optimal consensus method achieves significantly faster convergence than conventional approaches. Notably, this framework can be extended to tackle consensus problems in heterogeneous multi-agent systems.

1 | Introduction

The distributed cooperative control problem for multi-agent systems has recently garnered significant attention from various scientific communities. Through communication topologies, multiple agents can coordinate to solve tasks that are infeasible for a single agent, with applications spanning distributed robotics, unmanned aerial vehicles, wireless sensor networks, and satellite formation [1–4]. In this context, achieving consensus is a fundamental problem, requiring the design of distributed control protocols that guide all agents to agree on specific variables [5, 6]. As a result, numerous researchers have extensively studied the consensus problem from different perspectives.

In homogeneous multi-agent systems, all agents have identical system dynamics, with consensus typically categorized into

leaderless and leader-follower frameworks [7]. Early studies, such as [8], established a general framework for consensus in first-order networks with switching topologies, leveraging relative state information. Reference [9] later derived a less restrictive sufficient condition for consensus compared to [8]. Building on this, [10] extended first-order consensus protocols to distributed leader-following consensus for second-order integrator agents. Reference [11] investigated consensus in discrete-time systems, deriving an upper bound for the eigenratio and characterizing convergence rates. Due to the interplay between the nonzero eigenvalues of the graph Laplacian matrix and the feedback gain matrix, [12] introduced a coupling gain in their proposed distributed controller, proposing a decoupled method based on a continuous-time Riccati equation. Additionally, [13] investigated the synchronization of discrete-time multi-agent systems using a Riccati design-based approach. Recent works like [14–16] have

Abbreviations: ARE, algebraic Riccati equation; LQR, linear quadratic regulator.

proposed distributed adaptive control protocols. Notably, these approaches share a common limitation: consensus algorithms rarely optimize controller design, with convergence speed solely dependent on topology structure (Laplacian eigenvalues), which cannot be further optimized. Thus, designing optimal consensus controllers to enhance convergence speed and stability remains a critical challenge.

Although there have been recent advances in achieving optimal consensus for multi-agent systems, a major barrier to solving this problem is the requirement for full information from all agents in order to achieve global optimality. This poses a significant challenge in obtaining an analytical solution for the distributed controller. This fact was highlighted in 1968 when Witsenhausen presented a counterexample [17], demonstrating that the solution to linear optimal control problems with decentralized feedback control constraints may be a nonlinear function. Specifically, in networked multi-agent systems, when the optimal control theory is applied directly to solve the optimal consensus problem, due to the algebraic Riccati equations (AREs) coupled with the global information of the communication graph, the control input of each agent uses the information of its neighbors and non-neighbors. This leads to a centralized optimal controller rather than a distributed one. As mentioned in [18], the optimal topology is an all-to-all topology, which indicates there is a higher communication load among agents. On the other hand, [19] demonstrated that the optimal topology is a special star graph. Besides, [20] introduced a non-Riccati-based strategy to obtain a distributed optimal protocol by parameterizing the global cost function. However, this strategy is merely applicable to second-order dynamics. Thus, in order to address the challenge of solving an ARE for the linear multi-agent systems without topology information, [21] proposed an optimality criterion using the inverse optimal control method. This method introduced a highly complex performance index, while maintaining the simplicity of the Laplacian matrix for solving the optimal consensus problem. For additional research on distributed inverse optimal control, please refer to [22, 23]. In [24], a suboptimal LQ optimal design method was studied. Instead of optimizing a global criterion, the researchers focused on optimizing a local performance index. While [25–27] presented distributed optimal control protocols, the weighting matrices in the global objective function are required to follow a specific form that is linked to the Laplacian matrix of the communication graph. Furthermore, the symmetry requirements of the Laplacian matrix in [25, 27, 28] restrict the applicability of the optimal controller. Recently [29, 30], have proposed the distributed parallel optimization algorithm for the optimal consensus of multi-agent systems, where the communication topology among agents is an undirected graph and the obtained optimal controller is a numerical solution. Additionally, distributed optimal coordination algorithms for general multi-agent systems have been developed, whereby local states/outputs converge to the minimizing solution of a convex optimization problem. However, the requirement for local gradients to satisfy specific structures in [31], and the rank constraint condition on system matrices in [32, 33], still restrict the algorithm’s applicability. Overall, it is evident that the fundamental challenge of achieving optimal consensus in multi-agent systems remains unresolved.

Inspired by the analysis presented above, this paper aims to solve the optimal consensus problem for homogeneous systems with general linear dynamics and a cost function by proposing a novel approach based on the LQ optimal controller and distributed observers. The main contributions of this work are as follows:

1. We propose a novel consensus algorithm that minimizes the weighted state errors among neighboring agents over a directed graph. Specifically, we design an optimal consensus controller with an observer that incorporates each agent’s historical information, with the controller designed via solving an algebraic Riccati equation (ARE). The proposed optimal control is fully distributed, and relies solely on each agent’s relative state error information with respect to its neighbors, which is desirable but generally difficult to achieve.
2. Compared to the related works [21–27], our approach imposes no special requirements on the weight matrices in the global cost function. Notably, the developed control protocol achieves asymptotic optimality under the proposed framework, demonstrating greater flexibility and generality.
3. The optimal design of the distributed consensus protocol is equivalent to that of a decentralized controller, which can achieve a faster consensus speed compared to the traditional consensus method discussed in [11, 12]. Moreover, the proposed protocol eliminates the need to compute nonzero eigenvalues of the Laplacian matrix associated with the communication topology.

Notation. The following notations used in this paper are standard. $\mathbb{R}^{n \times m}$ represents the set of $n \times m$ -dimensional real matrices. I_N is the identity matrix of dimension $N \times N$. $\text{diag}\{a_1, a_2, \dots, a_N\}$ denotes the diagonal matrix with diagonal elements being a_1, \dots, a_N . $\rho(A)$ is the spectral radius of matrix A . $\lambda(A)$ is an eigenvalue of matrix A . $\|x\|$ is the 2-norm of a vector x . A^T denotes the transpose of a matrix A . \otimes denotes the Kronecker product.

The interaction network of multi-agent systems is characterized by a directed graph, represented as $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of vertices (nodes), $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the weight matrix of \mathcal{G} , $a_{ij} \neq 0$ if and only if the edge $(v_j, v_i) \in \mathcal{E}$, and we assume that the graph has no self-loop, that is, $a_{ii} = 0$. The neighbor of v_i is denoted by $\mathcal{N}_i = \{j | (v_j, v_i) \in \mathcal{E}\}$. The Laplacian matrix $\mathcal{L} = [l_{ij}]_{N \times N}$ associated with the adjacency matrix \mathcal{A} is defined as $l_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}$, $l_{ij} = -a_{ij}$ for $i \neq j$. A directed path from v_i to v_j is represented by a sequence of edges $(v_i, v_{i1}), (v_{i1}, v_{i2}), \dots, (v_{im}, v_j)$. If a node is reachable from every other node of the graph, then the graph contains a spanning tree.

2 | Preliminaries and Problem Statement

2.1 | Problem Statement

Consider a discrete-time homogeneous multi-agent system consisting of N agents on a general directed graph \mathcal{G} . The dynamics of each agent can be described as follows:

$$x_i(k+1) = Ax_i(k) + Bu_i(k), \quad i = 1, 2, \dots, N \quad (1)$$

where $x_i(k) \in \mathbb{R}^n$ is the state for agent i , and $u_i(k) \in \mathbb{R}^m$ is the input for agent i . $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are the coefficient matrices with an appropriate dimension.

Define the global cost function for multi-agent system (1) as

$$J(s, \infty) = \sum_{k=s}^{\infty} \left(\sum_{i=1}^N \sum_{j \in \mathcal{N}_i} (x_i(k) - x_j(k))^T Q (x_i(k) - x_j(k)) + \sum_{i=1}^N u_i^T(k) R_i u_i(k) \right) \quad (2)$$

where $Q \geq 0$ and $R_i > 0$ are weighting matrices.

Assumption 1. The directed communication topology \mathcal{G} has a spanning tree.

Problem 1. We seek a $u_i(k)$ for multi-agent system (1) that minimizes the cost function (2) based on the available information of an agent itself and its neighbors so that the closed-loop system of (1) can achieve state consensus, that is, for any initial conditions x_{i0} ,

$$\lim_{k \rightarrow \infty} \|x_i(k) - x_j(k)\| = 0, \quad i = 1, \dots, N \quad (3)$$

By virtue of the optimal control theory, as long as the optimal controller $u_i(k)$ exists, the condition (3) must be satisfied. In other words, when considering a multi-agent system (1), consensus is achieved and the protocol used is referred to as an optimal control-based protocol. It is important to note that this new proposed approach is completely distinct from classical approaches. In fact, the consensus protocol commonly used for multi-agent systems [8, 9] is formulated as:

$$u_i(k) = F \sum_{j \in \mathcal{N}_i} a_{ij} (x_j(k) - x_i(k)) \quad (4)$$

where F is a feedback gain matrix that depends on $\lambda_2(\mathcal{L})$ and $\lambda_N(\mathcal{L})$ as stated in [11]. Therefore, one needs to solve the non-zero eigenvalues for the Laplacian matrix \mathcal{L} , which requires the global information of the communication topology to determine the feedback gain F . Hence, the existing consensus algorithm (4) is not fully distributed.

Unlike widely used consensus protocols that are artificially defined, this article presents a fully distributed consensus protocol that is asymptotically optimal by optimizing a given LQ global performance. It is important to emphasize that the defined performance index (2) is more general, opposing the special case in existing results [21, 25–27].

2.2 | Preliminary

To solve Problem 1, we define the relative state error of the i th agent with respect to its neighbour agent j as

$$e_{ij}(k) = x_i(k) - x_j(k)$$

Then, it can be obtained from (1) that

$$e_{ij}(k+1) = Ae_{ij}(k) + Bu_i(k) - Bu_j(k) \quad (5)$$

Let $e_i(k) = [e_{ij_1}^T \ e_{ij_2}^T \ \dots \ e_{ij_{j_i}}^T]^T$ be the relative error vector between the i -th agent and its neighbor agent v with $v = j_1, \dots, j_{j_i}$. By stacking the error vectors, we can express the global error dynamics for the multi-agent system (1) in the following form:

$$\bar{e}(k+1) = \bar{A}\bar{e}(k) + \bar{B}u(k) \quad (6)$$

with $e(k) = [e_1^T(k) \ e_2^T(k) \ \dots \ e_N^T(k)]^T$ the global error vector, $u(k) = [u_1^T(k), \dots, u_N^T(k)]^T$, $\bar{A} = I_N \otimes A$, $\bar{B} = [\bar{B}_1 \ \dots \ \bar{B}_N]$, where \bar{B}_i consists of $0_{n \times m}$, B and $-B$. Meanwhile, the corresponding cost function (2) is rewritten as

$$J(s, \infty) = \sum_{k=s}^{\infty} \left(\sum_{i=1}^N \sum_{j \in \mathcal{N}_i} e_{ij}^T(k) Q e_{ij}(k) + \sum_{i=1}^N u_i^T(k) R_i u_i(k) \right) = \sum_{k=s}^{\infty} [e^T(k) \tilde{Q} e(k) + u^T(k) R u(k)] \quad (7)$$

with $\tilde{Q} = \text{diag}\{Q, Q, \dots, Q\} \geq 0$ and $R = \text{diag}\{R_1, R_2, \dots, R_N\} > 0$.

In the ideal case (a complete graph), all agents have access to the error information $e(k)$, the solvability of the optimal control problem for the system (6), with the global cost function (7), is equivalent to the classical standard LQ control problem. Additionally, by using the centralized optimal controller (8), the multi-agent system (1) can reach consensus.

Lemma 1 ([34]). *Let us assume that the relative error information $e(k)$ is available for all agents. Then the optimal controller with respect to the cost function (7) is expressed as*

$$u^*(k) = K_e e(k) \quad (8)$$

where the feedback gain K_e is given by

$$K_e = -(R + \bar{B}^T P_e \bar{B})^{-1} \bar{B}^T P_e \bar{A} \quad (9)$$

and P_e is the solution to the following ARE

$$P_e = \bar{A}^T P_e \bar{A} + \tilde{Q} - \bar{A}^T P_e \bar{B} (R + \bar{B}^T P_e \bar{B})^{-1} \bar{B}^T P_e \bar{A} \quad (10)$$

The corresponding optimal cost function is

$$J^*(s, \infty) = e^T(s) P_e e(s) \quad (11)$$

Moreover, if the ARE (10) has a unique positive definite solution P_e , then $\bar{A} + \bar{B} K_e$ is stable.

Unfortunately, when $e(k)$ is not available for all subsystems, the above classical LQR optimal design method in Lemma 1 can not be applicable directly to solve Problem 1. Therefore, this work introduces a novel distributed observer-based optimal controller framework.

3 | Distributed Optimal Controller Design

3.1 | Consensus of Multi-Agent System (1) Based on Relative Error Feedback

Let the relative error information of each agent with respect to its neighbors be exchanged in communication topology. Therefore, the optimal design of the distributed consensus controller of multi-agent systems (1) can be transformed into a decentralized controller, with the system (6) reformulated as follows:

$$e(k+1) = \bar{A}e(k) + \sum_{i=1}^N \bar{B}_i u_i(k) \quad (12)$$

$$\mathcal{Y}_i(k) = H_i e(k), \quad i = 1, \dots, N \quad (13)$$

where $\mathcal{Y}_i(k)$ is local measurement information, \bar{B}_i is the i th subpart of \bar{B} , H_i is a $\{0, I_n\}$ -matrix, whose specific forms rely on the interaction of agents.

First, we perform a row decomposition for the centralized optimal feedback gain matrix K_e given by (9),

$$K_e = \begin{bmatrix} K_{e1} \\ K_{e2} \\ \vdots \\ K_{eN} \end{bmatrix}$$

with $K_{ei} \in \mathbb{R}^{m \times Nn}$, and K_e is obtained by solving ARE (10).

To proceed further, the distributed feedback controllers is designed as

$$u_i^*(k) = K_{ei} \hat{e}_i(k), \quad i = 1, 2, \dots, N \quad (14)$$

where $\hat{e}_i(k)$ represents the distributed observer that estimate the global error vector $e(k)$ for each agent i , which is determined from the locally available information $\mathcal{Y}_i(k)$. The distributed observer $\hat{e}_i(k)$ is defined as

$$\begin{aligned} \hat{e}_1(k+1) &= \bar{A}\hat{e}_1(k) + \bar{B}_1 u_1^*(k) + \bar{B}_2 K_{e2} \hat{e}_1(k) \\ &+ \dots + \bar{B}_N K_{eN} \hat{e}_1(k) + \Upsilon_1(\mathcal{Y}_1(k) - H_1 \hat{e}_1(k)) \quad (15a) \\ &\dots \quad \dots \quad \dots \end{aligned}$$

$$\begin{aligned} \hat{e}_i(k+1) &= \bar{A}\hat{e}_i(k) + \bar{B}_1 K_{e1} \hat{e}_i(k) + \dots + \bar{B}_{i-1} K_{e_{i-1}} \hat{e}_i(k) \\ &+ \bar{B}_i u_i^*(k) + \bar{B}_{i+1} K_{e_{i+1}} \hat{e}_i(k) + \dots + \bar{B}_N K_{eN} \hat{e}_i(k) \\ &+ \Upsilon_i(\mathcal{Y}_i(k) - H_i \hat{e}_i(k)) \quad (15b) \\ &\dots \quad \dots \quad \dots \end{aligned}$$

$$\begin{aligned} \hat{e}_N(k+1) &= \bar{A}\hat{e}_N(k) + \bar{B}_1 K_{e1} \hat{e}_N(k) + \dots + \bar{B}_{N-1} K_{e_{N-1}} \hat{e}_N(k) \\ &+ \bar{B}_N u_N^*(k) + \Upsilon_N(\mathcal{Y}_N(k) - H_N \hat{e}_N(k)) \quad (15c) \end{aligned}$$

where the observer gain Υ_i will be determined later in order to guarantee the observer's stability.

Remark 1. Note that, since the control input $u_j(k)$ from non-neighbors is unknown for agent i , this poses a significant challenge in obtaining the analytical solution of the distributed

controller for the decentralized system (12). In this paper, a new type of distributed observer is proposed for the decentralized system (12), so $u_j(k)$ in (15) is replaced by $K_{ej} \hat{e}_i(k)$, where $\hat{e}_i(k)$ is designed for agent i based on the information of the agent and its neighbor to estimate the global error information $e(k)$. Based on this observer, a distributed optimal controller (14) is further developed, that is, $u_i^* = K_{ei} \hat{e}_i(k)$, which is different from the centralized form. More importantly, subsequent analysis reveals that the new proposed controller $u^*(k)$ asymptotically approaches the centralized optimal controller (8). In other words, the distributed controller (14) is asymptotically optimal. This implies that consensus can be achieved at a faster speed with the proposed control protocol.

Theorem 1. For the global error system (12), if the observer gain matrix $\Upsilon_i, i = 1, \dots, N$ exists such that the closed-loop matrix

$$\tilde{A}_{ec} = \begin{bmatrix} \Theta_1 & -\bar{B}_2 K_{e2} & \dots & -\bar{B}_N K_{eN} \\ -\bar{B}_1 K_{e1} & \Theta_2 & \dots & -\bar{B}_N K_{eN} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{B}_1 K_{e1} & \dots & -\bar{B}_{N-1} K_{e_{N-1}} & \Theta_N \end{bmatrix} \quad (16)$$

is stable, where $\Theta_i = \bar{A} + \bar{B}K_e - \bar{B}_i K_{ei} - \Upsilon_i H_i$, then the above observers (15) are stable under the distributed controller (14), that is,

$$\lim_{k \rightarrow \infty} \|\hat{e}_i(k) - e(k)\| = 0 \quad (17)$$

Moreover, if the Riccati equation (10) has a positive definite solution P_e , then the multi-agent system (1) can achieve consensus under the distributed feedback controllers(14).

Proof. Let the observer error vectors be denoted as:

$$\tilde{e}_i(k) = e(k) - \hat{e}_i(k) \quad (18)$$

Then, combining system (12) with observers (15) provides

$$\begin{aligned} e(k+1) &= (\bar{A} + \bar{B}K_e)e(k) - \bar{B}_1 K_{e1} \tilde{e}_1(k) \\ &- \bar{B}_2 K_{e2} \tilde{e}_2(k) - \dots - \bar{B}_N K_{eN} \tilde{e}_N(k) \quad (19) \end{aligned}$$

and the observer error systems are as follows

$$\begin{aligned} \tilde{e}_1(k+1) &= (\bar{A} + \bar{B}K_e - \bar{B}_1 K_{e1} - \Upsilon_1 H_1) \tilde{e}_1(k) \\ &- \bar{B}_2 K_{e2} \tilde{e}_2(k) - \dots - \bar{B}_N K_{eN} \tilde{e}_N(k) \quad (20a) \\ &\dots \quad \dots \quad \dots \end{aligned}$$

$$\begin{aligned} \tilde{e}_i(k+1) &= (\bar{A} + \bar{B}K_e - \bar{B}_i K_{ei} - \Upsilon_i H_i) \tilde{e}_i(k) \\ &- \bar{B}_1 K_{e1} \tilde{e}_1(k) - \dots - \bar{B}_{i-1} K_{e_{i-1}} \tilde{e}_{i-1}(k) \\ &- \bar{B}_{i+1} K_{e_{i+1}} \tilde{e}_{i+1}(k) - \dots - \bar{B}_N K_{eN} \tilde{e}_N(k) \quad (20b) \\ &\dots \quad \dots \quad \dots \end{aligned}$$

$$\begin{aligned} \tilde{e}_N(k+1) &= (\bar{A} + \bar{B}K_e - \bar{B}_N K_{eN} - \Upsilon_N H_N) \tilde{e}_N(k) \\ &- \bar{B}_1 K_{e1} \tilde{e}_1(k) - \dots - \bar{B}_{N-1} K_{e_{N-1}} \tilde{e}_{N-1}(k) \quad (20c) \end{aligned}$$

According to the observer error system (20), we derive the global observer error system

$$\tilde{e}(k+1) = \tilde{A}_{ec} \tilde{e}(k) \quad (21)$$

where $\tilde{e}(k) = [\tilde{e}_1^T(k), \tilde{e}_2^T(k), \dots, \tilde{e}_N^T(k)]^T$, and the matrix \tilde{A}_{ec} is the closed-loop system matrix. Therefore, if there exist gain matrices Υ_i such that \tilde{A}_{ec} is stable. Then, the observer error $\tilde{e}(k)$ can converge to origin as $k \rightarrow \infty$, that is, Equation (17) holds. In addition, on the basis of (19) and (20), we have

$$\begin{bmatrix} e(k+1) \\ \tilde{e}(k+1) \end{bmatrix} = \tilde{A}_{ec} \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix} \quad (22)$$

where $\tilde{A}_{ec} = \begin{bmatrix} \tilde{A} + \bar{B}K_e & \Omega_e \\ 0 & \tilde{A}_{ec} \end{bmatrix}$ and $\Omega_e = [-\bar{B}_1 K_{e1} \ \dots \ -\bar{B}_N K_{eN}]$.

Since P_e is the positive definite solution to ARE (10), $\tilde{A} + \bar{B}K_e$ is stable. With the aid of LQ control theory, the consensus of multi-agent system (1) can be achieved.

The proof is completed. \square

From Equation (22), and with the given value of K_e in equation (9), it is clear that the dynamics of the consensus error depend on \tilde{A}_{ec} , and the spectral radius value of \tilde{A}_{ec} is determined solely by the observer gains Υ_i . Therefore, in order to improve the convergence speed of the consensus, the remaining task is to find the optimal observer gain $\Upsilon_i, i = 1, 2, \dots, N$ such that the spectral radius of \tilde{A}_{ec} is minimized.

To this end, denote

$$A_e = \begin{bmatrix} A_K - \bar{B}_1 K_{e1} & -\bar{B}_2 K_{e2} & \dots & -\bar{B}_N K_{eN} \\ -\bar{B}_1 K_{e1} & A_K - \bar{B}_2 K_{e2} & \dots & -\bar{B}_N K_{eN} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{B}_1 K_{e1} & \dots & \dots & A_K - \bar{B}_N K_{eN} \end{bmatrix}$$

$$A_K = \tilde{A} + \bar{B}K_e, \quad H = \text{diag}\{H_1, H_2, \dots, H_N\}$$

$$U = \text{diag}\{U_1, U_2, \dots, U_N\}, \quad \Pi = \text{diag}\{\Pi_1, \Pi_2, \dots, \Pi_N\}$$

$$\Upsilon = \text{diag}\{\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N\}$$

Lemma 2. Assume that there exist matrices U, Π, S , and a parameter ω such that

$$S = S^T > 0 \quad (23a)$$

$$U = U^T > 0 \quad (23b)$$

$$\omega I - U > 0 \quad (23c)$$

$$\begin{bmatrix} U - S & (UA_e - \Pi H)^T \\ UA_e - \Pi H & U \end{bmatrix} > 0 \quad (23d)$$

and we select the observer gains $\Upsilon_i = U_i^{-1} \Pi_i$. Under the circumstances, the observer gain matrix can be obtained through solving the following optimization:

$$\min_{\Upsilon, U, S} \omega \quad \text{subject to:} \quad (23) \quad (24)$$

Proof. The observer error system (21) is rewritten as

$$\tilde{e}(k+1) = \tilde{A}_{ec} \tilde{e}(k) = (A_e - \Upsilon H) \tilde{e}(k) \quad (25)$$

The system (25) is asymptotically stable if there is a symmetric positive definite matrix U satisfying the Lyapunov inequality

$$(A_e - \Upsilon H)^T U (A_e - \Upsilon H) - U < 0 \quad (26)$$

Therefore, there exists a positive definite solution $S^T = S > 0$ such that

$$(A_e - \Upsilon H) U (A_e - \Upsilon H) - U < -S \quad (27)$$

Taking $\Upsilon = U^{-1} \Pi$, inequality (27) is replaced by (23d) equivalently. We introduce a quadratic cost function

$$J_\Upsilon = \sum_{k=0}^{T_N} \tilde{e}(k)^T S \tilde{e}(k) \quad (28)$$

By considering (27), we can derive

$$\begin{aligned} \tilde{e}(k+1)^T U \tilde{e}(k+1) - \tilde{e}(k)^T U \tilde{e}(k) &= \tilde{e}(k)^T (\tilde{A}_{ec}^T U \tilde{A}_{ec} - U) \tilde{e}(k) \\ &< -\tilde{e}(k)^T S \tilde{e}(k) \end{aligned}$$

Then,

$$J_\Upsilon \leq \tilde{e}^T(0) U \tilde{e}(0) - \tilde{e}^T(T_N) U \tilde{e}(T_N) \leq \tilde{e}^T(0) U \tilde{e}(0)$$

with $T_N \rightarrow \infty$, $\tilde{e}^T(T_N) U \tilde{e}(T_N) \rightarrow 0$. This inequality implies that the cost function (28) can be minimized by minimizing the bound $\tilde{e}^T(0) U \tilde{e}(0)$. Since $\tilde{e}^T(0) U \tilde{e}(0) \leq \|\tilde{e}(0)\|^2 \|U\| \leq M_0^2 \|U\|$, where M_0 denotes the upper bound of the initial error state $\tilde{e}(0)$, the optimal observer gain Υ is obtained by minimizing the maximum eigenvalue of U . In other words, we need to solve the minimization problem (24). \square

Solving the above optimization problem in Lemma 2 minimizes the estimator error of the observer gain. However, due to the constraints on the diagonal structure of U and Π , the derived observer gain Υ_i may be a suboptimal solution.

On the other hand, we will calculate the cost difference between the new proposed distributed controller (14) and the old centralized optimal control (8). After that, we will analyze the asymptotic optimality of the corresponding cost function.

For the purpose of analysis, denote

$$\begin{aligned} M_{e1} &= (\tilde{A} + \bar{B}K_e)^T P_e \Omega_e - \left[K_{e1}^T R_1 K_{e1} \ \dots \ K_{eN}^T R_N K_{eN} \right] \\ M_{e2} &= \begin{bmatrix} K_{e1}^T R_1 K_{e1} & 0 & \dots & 0 \\ 0 & K_{e2}^T R_2 K_{e2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & K_{eN}^T R_N K_{eN} \end{bmatrix} + \Omega_e^T P_e \Omega_e \\ \Omega_e &= \begin{bmatrix} -\bar{B}_1 K_{e1} & \dots & -\bar{B}_N K_{eN} \end{bmatrix} \end{aligned}$$

Theorem 2. Under the proposed distributed controllers (14) and (15) with Y_i chosen from the optimization in Lemma 2, the corresponding cost function is given by

$$J^*(s, \infty) = e^T(s)P_e e(s) + \sum_{k=s}^{\infty} \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix}^T \begin{bmatrix} 0 & M_{e1} \\ M_{e1}^T & M_{e2} \end{bmatrix} \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix} \quad (29)$$

Moreover, the cost difference between the cost function (29) and the cost under the centralized optimal control is expressed as

$$\begin{aligned} \Delta J(s, \infty) &= J^*(s, \infty) - J^*(s, \infty) \\ &= \sum_{k=s}^{\infty} \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix}^T \begin{bmatrix} 0 & M_{e1} \\ M_{e1}^T & M_{e2} \end{bmatrix} \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix} \end{aligned} \quad (30)$$

Notably, as s increases significantly, the optimal cost function difference will approach zero. Hence, the proposed consensus controller asymptotically achieves the optimal cost.

Proof. According to (10) and (19), we have

$$\begin{aligned} &e^T(k)P_e e(k) - e^T(k+1)P_e e(k+1) \\ &= e^T(k)(Q + K_e^T R K_e)e(k) \\ &\quad - \tilde{e}^T(k)\Omega_e^T P_e (\tilde{A} + \tilde{B}K_e)e(k) \\ &\quad - e^T(k)(\tilde{A} + \tilde{B}K_e)^T P_e \Omega_e \tilde{e}(k) - \tilde{e}^T(k)\Omega_e^T P_e \Omega_e \tilde{e}(k) \end{aligned}$$

Based on the cost function (7) for centralized optimal control, we can perform summation on k from s to ∞ and apply algebraic calculations to simplify the expression

$$\begin{aligned} &e^T(s)P_e e(s) - e^T(\infty)P_e e(\infty) \\ &= J(s, \infty) - \sum_{k=s}^{\infty} \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix}^T \begin{bmatrix} 0 & M_{e1} \\ M_{e1}^T & M_{e2} \end{bmatrix} \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix} \end{aligned} \quad (31)$$

It follows from Theorem 1 that $\lim_{k \rightarrow \infty} e^T(k)P_e e(k) = 0$. Therefore, the corresponding optimal cost function $J^*(s, \infty)$ under the proposed distributed observer-based controller (14) is derived in (29). Furthermore, in accordance with equation (11), the optimal cost difference stated in equation (30) is valid.

As stated in Theorem 1, the closed-loop system given by equation (22) is stable. Therefore, there exist two constants, namely $a > 0$ and $0 < \gamma < 1$, that satisfy the following condition:

$$\left\| \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix} \right\| \leq a\gamma^k \left\| \begin{bmatrix} e(0) \\ \tilde{e}(0) \end{bmatrix} \right\| \quad (32)$$

Then, we have

$$\begin{aligned} \Delta J_e(s, \infty) &= \sum_{k=s}^{\infty} \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix}^T \begin{bmatrix} 0 & M_{e1} \\ M_{e1}^T & M_{e2} \end{bmatrix} \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix} \\ &\leq \sum_{k=s}^{\infty} \left\| \begin{bmatrix} 0 & M_{e1} \\ M_{e1}^T & M_{e2} \end{bmatrix} \right\| \left\| \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix} \right\|^2 \end{aligned}$$

$$\begin{aligned} &\leq \left\| \begin{bmatrix} 0 & M_{e1} \\ M_{e1}^T & M_{e2} \end{bmatrix} \right\| \left\| \begin{bmatrix} e(0) \\ \tilde{e}(0) \end{bmatrix} \right\|^2 \sum_{k=s}^{\infty} a^2 \gamma^{2k} \\ &= \bar{a}\gamma^{2s} \end{aligned} \quad (33)$$

$$\text{with } \bar{a} = \left\| \begin{bmatrix} 0 & M_{e1} \\ M_{e1}^T & M_{e2} \end{bmatrix} \right\| \left\| \begin{bmatrix} e(0) \\ \tilde{e}(0) \end{bmatrix} \right\|^2 \frac{a^2}{1-\gamma^2}.$$

Since $0 < \gamma < 1$, for any given $\varepsilon > 0$, there exists a sufficiently large integer M such that $\gamma^{2M} < \frac{\varepsilon}{\bar{a}}$. Based on (33), it holds that

$$\sum_{k=M}^{\infty} \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix}^T \begin{bmatrix} 0 & M_{e1} \\ M_{e1}^T & M_{e2} \end{bmatrix} \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix} < \varepsilon \quad (34)$$

It can be seen that when M is large enough, the cost difference (30) satisfies the following inequality:

$$\Delta J(M, \infty) < \varepsilon$$

Thus, the optimal cost difference (30) equals 0 as $s \rightarrow \infty$. \square

3.2 | Comparison to Traditional Consensus Algorithms

In this subsection, we will compare our consensus algorithm with traditional ones to highlight its advantages and improvements.

Firstly, let's discuss the aspect of convergence performance. The consensus achieved under the new controller exhibits a faster convergence speed compared to the traditional consensus algorithm, which typically relies on gradient descent and shows a linear convergence rate.

In fact, under the new distributed control protocol (14), an asymptotic convergence factor is introduced to measure the convergence speed for consensus [35]

$$r_{\text{asym}} = \sup_{e(0), \tilde{e}(0) \neq 0} \lim_{k \rightarrow \infty} \left(\frac{\left\| \begin{bmatrix} e(k) \\ \tilde{e}(k) \end{bmatrix} \right\|}{\left\| \begin{bmatrix} e(0) \\ \tilde{e}(0) \end{bmatrix} \right\|} \right)^{1/k} \quad (35)$$

For the closed-loop system (22), the asymptotic convergence factor r_{asym} is evaluated as follows:

$$r_{\text{asym}} = \rho(\tilde{A}_{ec}) \quad (36)$$

which implicates that the convergence speed of the distributed asymptotically optimal consensus algorithm (14) is characterized by the spectra radius of \tilde{A}_{ec} that depends on $\rho(\tilde{A} + \tilde{B}K_e)$ and $\rho(\tilde{A}_{ec})$. Therefore, for a faster convergence speed, the control gain K_e and observer gain Y_i should be carefully selected to minimize r_{asym} or $\rho(\tilde{A}_{ec})$. In particular, $\tilde{A} + \tilde{B}K_e$ represents the closed-loop system matrix obtained through the optimal feedback control (9). This means that $e(k+1) = (\tilde{A} + \tilde{B}K_e)e(k)$, and the minimization of $e(k+1)^T \tilde{Q}e(k+1)$ is done according to (7), with the goal of minimizing the modulus of the eigenvalues for $\tilde{A} + \tilde{B}K_e$.

Additionally, by optimizing (24), one can appropriately select Y_i in order to minimize the upper bound of the spectral radius $\rho(\tilde{A}_{ec})$. In this way, $\rho(\tilde{A}_{ec})$ is made small. This approach differs from conventional consensus algorithms, where the maximum eigenvalue of the closed-loop system matrix \tilde{A}_{ec} is not minimized and is determined by the eigenvalues of the Laplacian matrix \mathcal{L} , that is to say, the traditional convergence rate depends on the non-zero eigenvalues $\lambda_i(\mathcal{L})$. The faster convergence performance is also demonstrated in the simulation example in Section 4.

Secondly, our consensus algorithm is asymptotically optimal. The cost difference $\Delta J(s, \infty)$ between the new distributed controller (14) and the centralized controller (8) is explained in Theorem 2, and it approaches zero as s tends to infinity.

Thirdly, the cost function is quite general, which is independent of the global topology information and the weight matrix requires no special construction.

3.3 | Special Case: Consensus of Multi-Agent Systems (1) via State Feedback Controller

Different from Section 3.1, another information exchange strategy in multi-agent networks involves each agent exchanging state information with its neighbors.

By stacking the state vectors, we can rewrite the multi-agent system (1) as follows:

$$X(k+1) = \tilde{A}X(k) + \tilde{B}u(k) \quad (37)$$

where $X(k) = [x_1^T(k), \dots, x_N^T(k)]^T$ is the global state variable. $\tilde{A} = I_N \otimes A$ and $\tilde{B} = \text{diag}\{B_1, B_2, \dots, B_N\}$.

Besides, the cost function (2) is rewritten as

$$J(s, \infty) = \sum_{k=s}^{\infty} [X^T(k)QX(k) + u^T(k)Ru(k)] \quad (38)$$

where $Q = [Q]_{ij} \geq 0$ with $Q_{ii} = (N-1)Q$, $Q_{ij} = -Q$ for $i \neq j$, and $R = \text{diag}\{R_1, R_2, \dots, R_N\} > 0$.

Lemma 3. Assume the state information $X(k)$ is available for all agents subject to (1). Then, the optimal controller with respect to the cost function (2) is given by:

$$u^*(k) = KX(k) \quad (39)$$

where the feedback gain matrix $K = \mathcal{L} \otimes F$ is given as

$$K = -(R + \tilde{B}^T P \tilde{B})^{-1} \tilde{B}^T P \tilde{A} \quad (40)$$

and P is the solution to the following ARE

$$P = \tilde{A}^T P \tilde{A} + Q - \tilde{A}^T P \tilde{B} (R + \tilde{B}^T P \tilde{B})^{-1} \tilde{B}^T P \tilde{A} \quad (41)$$

The corresponding optimal cost function is

$$J^*(s, \infty) = X^T(s)PX(s) \quad (42)$$

Moreover, if P is the unique positive definite solution to (41), then $\tilde{A} + \tilde{B}K$ is stable.

According to Lemma 3, the centralized optimal controller (39) requires an all-to-all communication network, which obviously has limited practicability. Therefore, this work designs a new distributed observer type for each agent i to estimate the global state $X(k)$ by only employing the information of itself and its neighbors. On this basis, we develop a distributed state feedback controller for multi-agent systems (1) that achieves consensus.

System (37) is rewritten as

$$X(k+1) = \tilde{A}X(k) + \sum_{i=1}^N \tilde{B}_i u_i(k) \quad (43)$$

$$Y_i(k) = C_i X(k) \quad (44)$$

where $Y_i(k)$ is available measurement information, $\tilde{B}_i = [0 \dots B^T 0 \dots 0]^T$, C_i is a $\{0, I_n\}$ -matrix determined by communication interaction among agents.

Moreover, we design a distributed observer-based state feedback controller for each agent i :

$$u_i^*(k) = K_i \hat{X}_i(k), \quad i = 1, 2, \dots, N \quad (45)$$

with the distributed observers $\hat{X}_i(k)$ given by

$$\begin{aligned} \hat{X}_i(k+1) = & \tilde{A} \hat{X}_i(k) + \tilde{B}_1 K_1 \hat{X}_1(k) + \dots + \tilde{B}_{i-1} K_{i-1} \hat{X}_{i-1}(k) + \tilde{B}_i u_i^*(k) \\ & + \dots + \tilde{B}_N K_N \hat{X}_N(k) + L_i (Y_i(k) - C_i \hat{X}_i(k)) \end{aligned} \quad (46)$$

where L_i is the observer gain, and $K_i = [0 \ 0 \ \dots \ I \ \dots \ 0]K$ is the subpart of matrix K obtained by solving ARE (41).

Theorem 3. Let Assumption 1 hold. Consider the global form (43) of the multi-agent system (1), and the control law (45). If there exist observer gains L_i such that the matrix

$$\tilde{A}_c = \begin{bmatrix} W_1 & -\tilde{B}_2 K_2 & \dots & -\tilde{B}_N K_N \\ -\tilde{B}_1 K_1 & W_2 & \dots & -\tilde{B}_N K_N \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{B}_1 K_1 & -\tilde{B}_2 K_2 & \dots & W_N \end{bmatrix} \quad (47)$$

is stable with $W_i = \tilde{A} + \tilde{B}K - \tilde{B}_i K_i - L_i C_i$, then the observers (46) are stable, that is,

$$\lim_{k \rightarrow \infty} \|\hat{X}_i(k) - X(k)\| = 0, \quad i = 1, \dots, N \quad (48)$$

Moreover, if P is the positive definite solution of (41), under the feedback controller (45), the multi-agent system (1) can achieve consensus.

Proof. The proof is similar to Theorem 1, and is therefore neglected. \square

Through the analysis of Theorem 3, it has been shown that consensus is achieved among all agents under the distributed state feedback controllers (45) and (46). Furthermore, the state of each

agent converges to zero, which aligns with the result obtained for a single system [36].

Similarly, the theorem also discusses the asymptotical optimal property of the newly proposed distributed state feedback controllers (45), as presented in Theorem 2.

Theorem 4. Under the proposed distributed controller (45) and (46) with $L_i, i = 1, 2, \dots, N$ chosen from Theorem 3, the cost function is given by

$$J^*(s, \infty) = X^T(s)PX(s) + \sum_{k=s}^{\infty} \begin{bmatrix} X(k) \\ \tilde{X}(k) \end{bmatrix}^T \begin{bmatrix} 0 & M_1 \\ M_1^T & M_2 \end{bmatrix} \begin{bmatrix} X(k) \\ \tilde{X}(k) \end{bmatrix} \quad (49)$$

where

$$M_1 = (\tilde{A} + \tilde{B}K)^T P\Omega - \begin{bmatrix} K_1^T R_1 K_1 & \dots & K_N^T R_N K_N \end{bmatrix}$$

$$M_2 = \begin{bmatrix} K_1^T R_1 K_1 & 0 & \dots & 0 \\ 0 & K_2^T R_2 K_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & K_N^T R_N K_N \end{bmatrix} + \Omega^T P\Omega$$

$$\Omega = \begin{bmatrix} -\tilde{B}_1 K_1 & \dots & -\tilde{B}_N K_N \end{bmatrix}$$

Moreover, the cost difference between the cost function (49) and the optimal cost (42) is given by

$$\Delta J(s, \infty) = J^*(s, \infty) - J^*(s, \infty)$$

$$= \sum_{k=s}^{\infty} \begin{bmatrix} X(k) \\ \tilde{X}(k) \end{bmatrix}^T \begin{bmatrix} 0 & M_1 \\ M_1^T & M_2 \end{bmatrix} \begin{bmatrix} X(k) \\ \tilde{X}(k) \end{bmatrix} \quad (50)$$

Specifically, when s is large enough, the cost difference will be zero. This means that the proposed consensus controller (45) asymptotically achieves the optimal cost.

Proof. This proof is similar to that in Theorem 2. So the details are neglected. \square

4 | Numerical Simulation

In this section, two numerical examples are implemented to indicate the effectiveness of the proposed new theoretical results.

Example 1. Consider the multi-agent system composed of four homogeneous agents. The system matrices for these agents are taken from [37],

$$A = \begin{bmatrix} 1.1 & 0.3 \\ 0 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad (51)$$

Figure 1 presents the agents' interactions, where each agent receives neighbor error information. Under this communication interaction, $H_i, i = 1, 2, 3, 4$ are determined as:

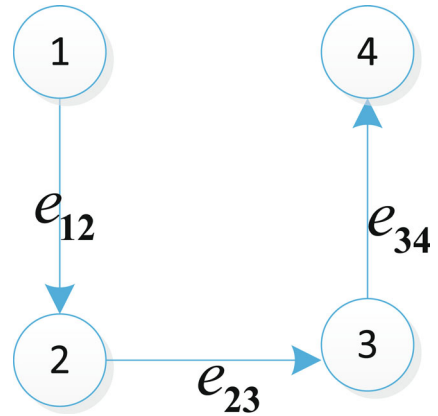


FIGURE 1 | Communication topology among four agents.

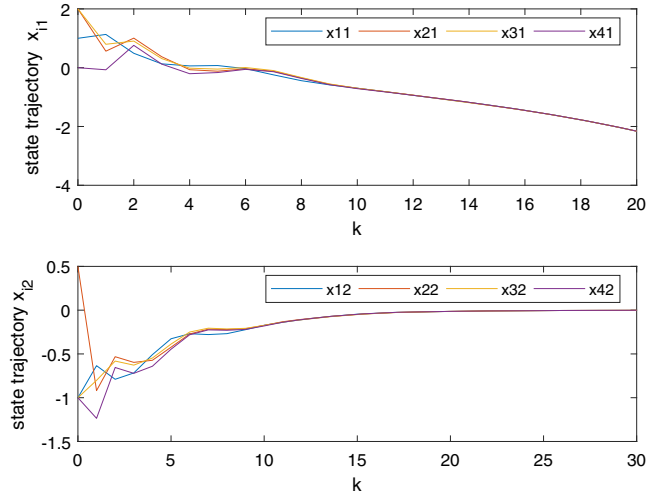


FIGURE 2 | The state trajectories for each agent $x_i(k), i = 1, 2, 3, 4$.

$$H_1 = \begin{bmatrix} I_2 & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix}, H_4 = \begin{bmatrix} 0 & 0 & I_2 \end{bmatrix}$$

We choose

$$Q = I_2, R_1 = R_2 = R_3 = R_4 = 1$$

According to ARE (10) and the optimization solution in (24), we can obtain the feedback gain K_{ei} and the observer gain. Figure 2 illustrates the evolutionary trend of each agent's state using the new proposed consensus algorithm (referred to as M2), and highlights that the state of all agents reaches the consensus value within 12 steps. Figure 3 shows that the error vector of the observer, $e_1(k) - \hat{e}_1(k)$, for agent 1 converges to zero. This indicates that the distributed observer is stable. Under the same initial conditions, Figure 4 depicts the state trajectories by using the traditional consensus method (referred to as M1) [8], highlighting that the second state of each agent takes a longer time to reach

a common state. Table 1 presents the quantitative calculations, which are based on the spectral radius $\rho(\tilde{A}_{ec})$, as well as the norm of the first agent's state at various time intervals. The corresponding results infer that the proposed consensus algorithm reduces the maximum eigenvalue of \tilde{A}_c and the norm of each agent's state $\|x_i(0)\|$. Figures 5 and 6 show the absolute value of consensus error $|x_{i1} - x_{11}|$ by the new method and the traditional consensus method, respectively. It's clear that the magnitude of the consensus error is much smaller than that of the traditional distributed

control law. As a result, the proposed distributed observer-based consensus algorithm (14) guarantees faster convergence speed for all agents to reach consensus, and a smaller magnitude of consensus error.

Example 2. In this example, we consider the directed communication graph of three agents described in Figure 7. The system matrix of each agent is given by

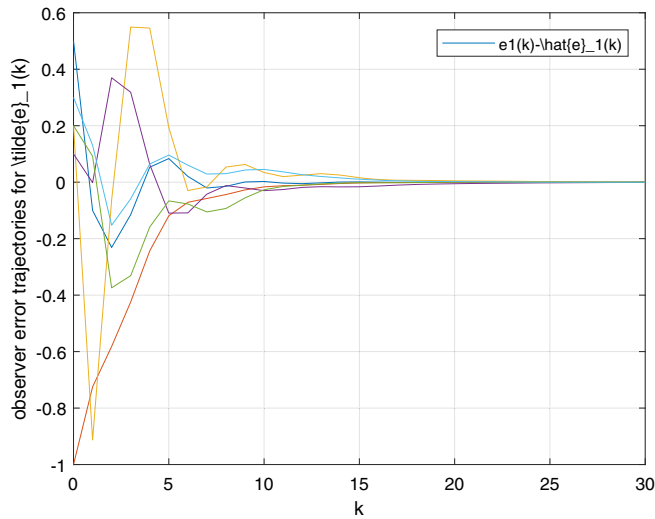


FIGURE 3 | Observer error trajectories $\tilde{e}_1(k)$.

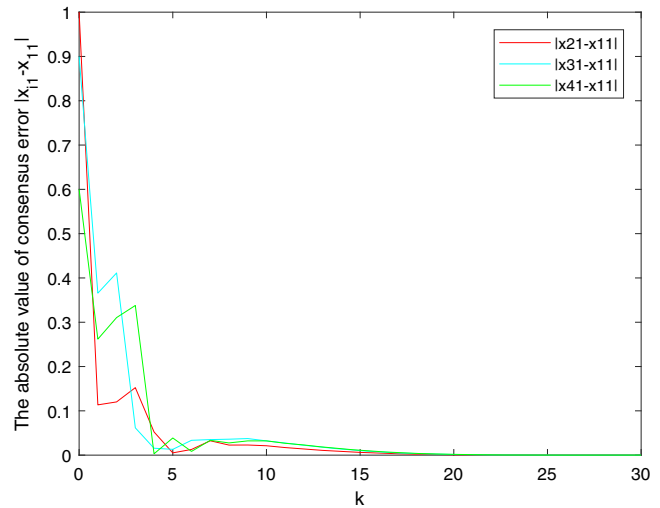


FIGURE 5 | The absolute value of consensus error $|x_{i1} - x_{11}|$ by M2.

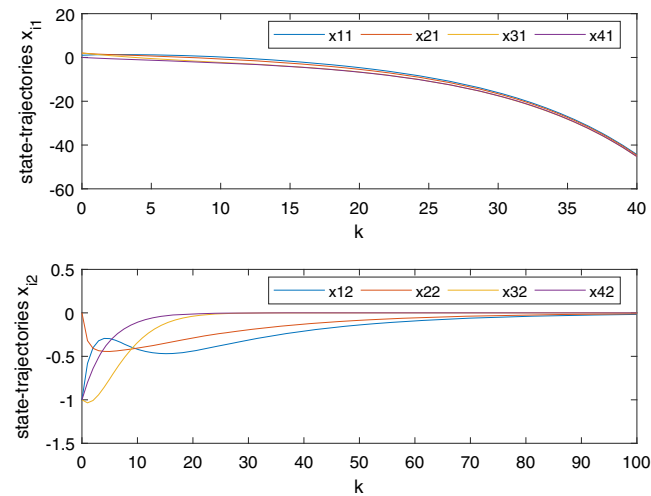


FIGURE 4 | The state trajectories $x_i(k)$ by the traditional consensus method.

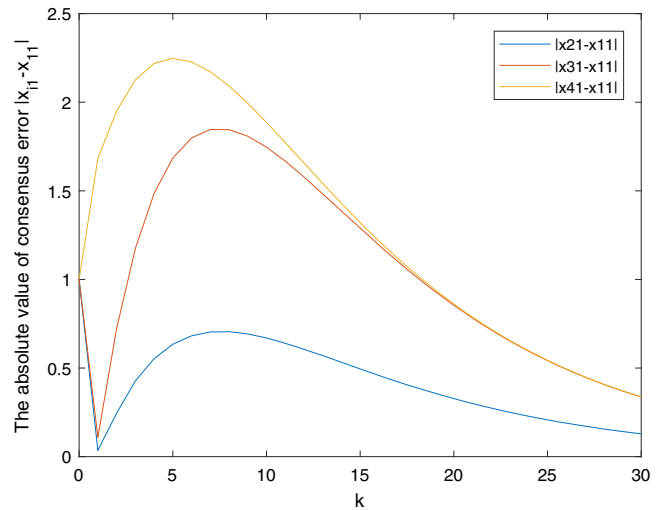


FIGURE 6 | The absolute value of consensus error $|x_{i1} - x_{11}|$ by M1.

TABLE 1 | $\rho(\tilde{A}_{ec})$ and the norm of the first agent's state by using different algorithms.

Method	$\rho(\tilde{A}_{ec})$	$\ x_1\ _2$										
		Step1	Step2	Step4	Step6	Step8	Step10	Step12	Step14	Step16	Step18	Step20
M1	0.8777	1.4142	1.3670	1.3639	1.2424	0.9817	0.6218	0.4494	0.9513	1.7692	2.7805	3.9881
M2	0.7297	1.4142	1.2972	0.7293	0.3349	0.3702	0.6243	0.8271	1.0638	1.3096	1.6056	1.9587

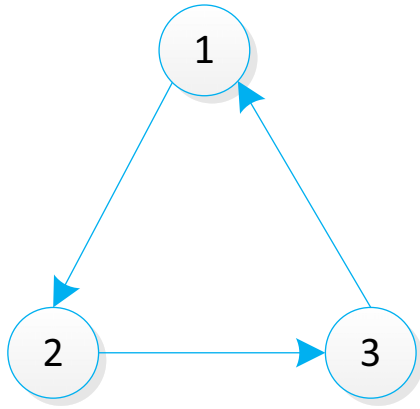


FIGURE 7 | A directed topology among three agents.

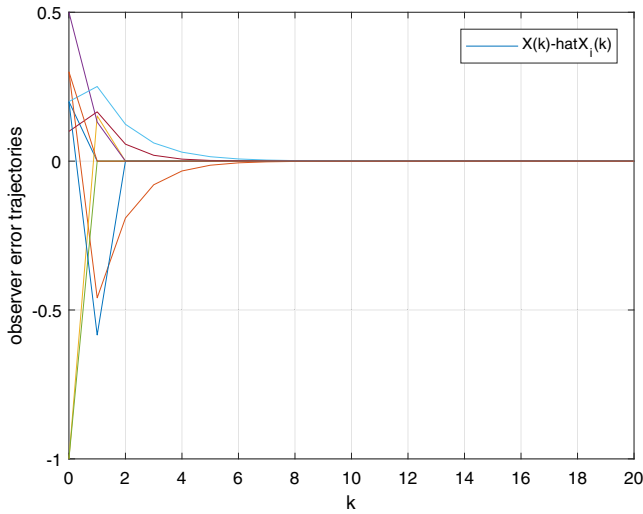


FIGURE 8 | Observer error trajectories $\tilde{X}_i(k)$ in Theorem 3.

$$A = 1, B = \begin{bmatrix} 1.5 & 0.5 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, C_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q = 1, R_1 = R_2 = R_3 = 1$$

The feedback gains in (40) are obtained by the solution of ARE (41) as

$$K_1 = \begin{bmatrix} -0.3935 & 0 & 0 \\ -0.1312 & 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & -0.3935 & 0 \\ 0 & -0.1312 & 0 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 0 & 0 & -0.3935 \\ 0 & 0 & -0.1312 \end{bmatrix}$$

According to Theorem 3 and the solution of the optimization problem in (24), we get the observer gain matrices as:

$$L_1 = \begin{bmatrix} 1.0000 & 0 \\ 0 & 0 \\ 0 & 0.4934 \end{bmatrix}, L_2 = \begin{bmatrix} 0.3441 & 0 \\ 0 & 1.0000 \\ 0 & 0 \end{bmatrix},$$

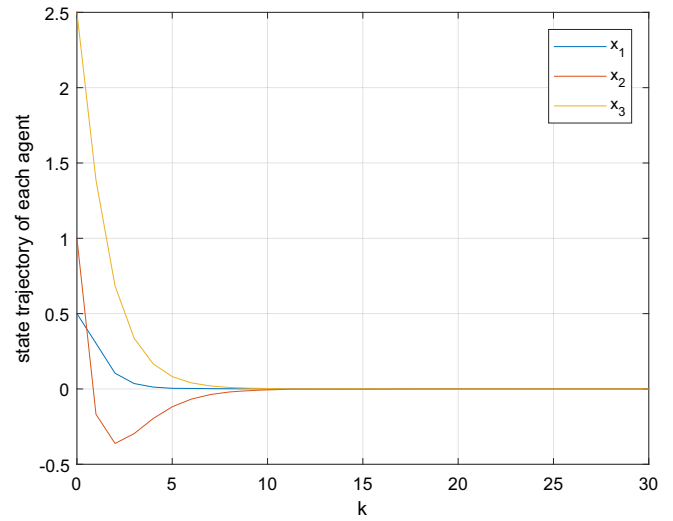


FIGURE 9 | The state trajectory of each agent $x_i(k), i = 1, 2, 3$.

$$L_3 = \begin{bmatrix} 0 & 0 \\ 0.3441 & 0 \\ 0 & 1.0000 \end{bmatrix}$$

From Figure 8, it can be observed that the observer error systems are stable. Meanwhile, Figure 9 shows that the states of all agents converge to zero within ten steps under the distributed state feedback controllers (45). These results clearly demonstrate the effectiveness of the proposed distributed controller.

5 | Conclusions

This article solved the optimal consensus problem for the general case based on the LQ optimal controller and observer. To the best of our knowledge, this achievement marks the first time a novel distributed controller has been designed using observers that incorporate the agent's historical state information by solving Riccati equations. Unlike existing consensus algorithms, this new approach does not require the calculation of eigenvalues for the communication topology. It significantly enhances consensus speed compared to traditional methods. Moreover, the proposed consensus algorithm can solve the output consensus for the heterogeneous systems without requiring reference model, the relevant works can refer to [38] and [39]. Besides, the derived observer gain Υ is a suboptimal solution in Lemma 2 with a special structure, and the system dynamic of each agent does not consider the effect of measure noise. Therefore, future work will be devoted to provide the solvability of the optimal observer gain and study the robustness of the proposed controller against measure noise.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Research data are not shared.

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